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SECOND ORDER NON VARIATIONAL BASIC PARABOLIC SYSTEMS

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Dedicated to Professor Francesco Guglielmino
with our deepest esteem and gratitude, on his 70th birthday

Let Q be the cylinder $\Omega \times (-T, 0)$ and $W^p(Q, \mathbb{R}^k)$ ($p \geq 1$, k integer ≥ 1) the Banach space

$$W^p(Q, \mathbb{R}^k) = \{v : v \in L^p(-T, 0, H^{2,p}(\Omega, \mathbb{R}^k)), \frac{\partial v}{\partial t} \in L^p(Q, \mathbb{R}^k)\};$$

if $u \in W^2(Q, \mathbb{R}^N)$ (N integer ≥ 1) is a solution in Q of the basic system

$$a(H(u)) - \frac{\partial u}{\partial t} = 0,$$

where $a(\xi)$ is a vector of $\mathbb{R}^N$, continuous onto $\mathbb{R}^{n^2 N}$, satisfying the conditions $a(0) = 0$ and (A), we show that $Du \in W^q_{\text{loc}}(Q, \mathbb{R}^{nN})$ with $q > 2$ and we derive the so called fundamental estimates for the matrix $H(u)$ and the vector $\frac{\partial u}{\partial t}$. In a standard way, from the fundamental estimates, we deduce

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that Du and u are Hölder-continuous in Q, if $n = 2$ and if $n \leq 4$, respectively. Moreover we study the Hölder-continuity in Q of the vectors Du and u, when u is a solution of the system:

$$a(H(u)) - \frac{\partial u}{\partial t} = f(X), \quad f \in \mathscr{L}^{2,\mu}(Q, \mathbb{R}^N),$$

and also we give a first result of Hölder-continuity in Q for the solutions of the system:

$$a(H(u)) - \frac{\partial u}{\partial t} = b(X, u, Du),$$

with b vector of $\mathbb{R}^N$ with "linear growth".

1. Introduction.

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, of class C^2, with generic point $x = (x_1, x_2, \dots, x_n)$. If T is a real positive number, we denote by Q the cylinder $\Omega \times (-T, 0)$ and by X the point (x, t) of $\mathbb{R}^n_x \times \mathbb{R}_t$. If $u(X)$ is a vector $Q \to \mathbb{R}^N$, N integer ≥ 1, we set:

$$D_i u = \frac{\partial u}{\partial x_i}, \quad Du = (D_1 u, D_2 u, \dots, D_n u),$$

$$H(u) = \{D_i D_j u\} = \{D_{ij} u\}, \quad i, j = 1, 2, \dots, n;$$

Du and $H(u)$ are elements of $\mathbb{R}^{nN}$ and $\mathbb{R}^{n^2 N}$, respectively.
Setting

$$W^p(Q, \mathbb{R}^k) = \left\{ v : v \in L^p(-T, 0, H^{2,p}(\Omega, \mathbb{R}^k)), \ \frac{\partial v}{\partial t} \in L^p(Q, \mathbb{R}^k) \right\},$$

$$W^p_0(Q, \mathbb{R}^k) = \left\{ v \in W^p(Q, \mathbb{R}^k) : v \in L^p(-T, 0, H^{1,p}_0(\Omega, \mathbb{R}^k)), \right.$$

$$\left. v(x, -T) = 0 \right\},$$

where $p \in [1, +\infty[$, k is an integer ≥ 1 and $H^{2,p}(\Omega, \mathbb{R}^k)$, $H^{1,p}_0(\Omega, \mathbb{R}^k)$ are the usual Sobolev spaces ([1]), let $u \in W^2(Q, \mathbb{R}^N)$ be a solution in Q of the basic

([1]) $W^p(Q, \mathbb{R}^k)$ and $W^p_0(Q, \mathbb{R}^k)$ are Banach spaces provided by the norm

$$\|u\|_{p,Q} = \left[\int_Q (\|u\|^p + \|Du\|^p + \|H(u)\|^p + \|\frac{\partial u}{\partial t}\|^p) \, dX \right]^{\frac{1}{p}}.$$

system:

$$a\big(H(u)\big) - \frac{\partial u}{\partial t} = 0, \tag{1.1}$$

where $a(\xi)$ is a vector of $\mathbb{R}^N$, continuous onto $\mathbb{R}^{n^2N}$, satisfying the conditions:

$$a(0) = 0; \tag{1.2}$$

(A) *there exist three positive constants* α, γ *and* δ *with* $\gamma + \delta < 1$, *such that:*

$$\Big\| \sum_{i=1}^{n} \tau_{ii} - \alpha\big[a(\tau+\eta) - a(\eta)\big]\Big\|^2 \leq \gamma\|\tau\|^2 + \delta\Big\|\sum_{i=1}^{n}\tau_{ii}\Big\|^2, \quad \forall\, \tau, \eta \in \mathbb{R}^{n^2N}.$$

From the condition (A), setting $\eta = 0$, we get, $\forall\, \tau \in \mathbb{R}^{n^2N}$

$$\|a(\tau)\| \leq \frac{c(n)}{\alpha}\|\tau\|. \tag{1.3}$$

In Section 2 we shall prove, by a technique similar to the that one used by S. Campanato [2] in the elliptic case (see also [3]), the following result of differentiability:

Theorem 1.1. *If the vector* $a(\xi)$ *satisfies the conditions* (1.2) *and* (A)*, then*

$$Du \in W^2_{\mathrm{loc}}(Q, \mathbb{R}^{nN}) \tag{1.4}$$

and, $\forall\, Q(2\sigma) = Q(X^0, 2\sigma) = B(x^0, 2\sigma) \times (t^0 - (2\sigma)^2, t^0) \subset Q$*, the following Caccioppoli's type estimate holds:*

$$\int_{Q(\sigma)} \Big(\|H(Du)\|^2 + \|\frac{\partial(Du)}{\partial t}\|^2\Big)\, dX \leq \tag{1.5}$$

$$\leq c\sigma^{-2}\Big\{\sigma^{-2}\int_{Q(2\sigma)}\|D(u - P_{Q(2\sigma)})\|^2 dX + \int_{Q(2\sigma)}\|H(u - P_{Q(2\sigma)})\|^2 dX\Big\},$$

where the constant c does not depend on σ *and* $P_{Q(2\sigma)}$ *is the vector-polynomial in* x*, of degree* ≤ 2*, such that*

$$\int_{Q(2\sigma)} D^{\alpha}(u - P_{Q(2\sigma)})dX = 0, \quad \forall \alpha : |\alpha| \leq 2\ (^2). \tag{1.6}$$

(^{2}) $D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, α_i integer ≥ 0.

From this result, in virtue of the well known Gehring-Giaquinta-G.Modica Lemma, the L^q_{loc}-regularity of the vectors $H(Du)$ and $\frac{\partial(Du)}{\partial t}$ will be derived. Moreover we shall prove the following existence and uniqueness result:

Theorem 1.2. *If Ω is of class C^2 and convex and if the vector $a(\xi)$ satisfies the conditions* (1.2) *and* (A)*, then, $\forall \varphi \in L^2(Q, \mathbb{R}^N)$ and $\forall u \in W^2(Q, \mathbb{R}^N)$, the Cauchy-Dirichlet problem:*

$$\text{(1.7)} \qquad \begin{cases} w \in W^2_0(Q, \mathbb{R}^N) \\ a\big(H(w) + H(u)\big) - \dfrac{\partial w}{\partial t} = \varphi(X) \ \ in \ \ Q \end{cases}$$

has a unique solution. Moreover the following estimate holds:

$$\text{(1.8)} \qquad \int_Q \left(\|H(w)\|^2 + \left\| \frac{\partial w}{\partial t} \right\|^2 \right) dX \leq c(\alpha, \gamma, \delta) \int_Q \left\| \varphi - a\big(H(u)\big) \right\|^2 dX.$$

In Section 3 we will give the interior fundamental estimates for $H(Du)$, $\frac{\partial(Du)}{\partial t}$, $H(u)$ and $\frac{\partial u}{\partial t}$ which will enable us to achieve the Hölder-continuity in Q of Du and u, if $n = 2$ and if $n \leq 4$, respectively. Thus we obtain, following a different method, the same results obtained by S. Campanato in the Section 5 of [3]. Moreover we will show that the solutions $u \in W^2(Q, \mathbb{R}^N)$ of the system

$$\text{(1.9)} \qquad a\big(H(u)\big) - \frac{\partial u}{\partial t} = f(X), \quad f \in \mathcal{L}^{2,\mu}(Q, \mathbb{R}^N),$$

are Hölder-continuous in Q if $n \leq 4$ and in Section 5 we will study the Hölder-continuity in Q of the solutions $u \in W^2(Q, \mathbb{R}^N)$ of the system

$$\text{(1.10)} \qquad a\big(H(u)\big) - \frac{\partial u}{\partial t} = b(X, u, Du),$$

with $b(X, u, p)$ vector of $\mathbb{R}^N$ with "linear growth".

2. Proof of Theorems 1.1 and 1.2 and L^q_{loc}-regularity.

Let $u \in W^2(Q, \mathbb{R}^N)$ be a solution in Q of the basic system

$$\text{(2.1)} \qquad a\big(H(u)\big) - \frac{\partial u}{\partial t} = 0,$$

where $a(\xi)$ is a vector of $\mathbb{R}^N$, continuous onto $\mathbb{R}^{n^2N}$, satisfying the conditions (1.2) and (A).

Fixed the cylinder $Q(2\sigma) = Q(X^0, 2\sigma) \subset Q$, let $\vartheta(x)$ and $g(t)$ be two real functions of class $C_0^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R})$ respectively, satisfying the following properties:

$$0 \le \vartheta \le 1,\ \vartheta = 1 \text{ in } B(x^0, \sigma),\ \vartheta = 0 \text{ in } \mathbb{R}^n \setminus B(x^0, \frac{3}{2}\sigma), \tag{2.2}$$

$$|D^\alpha \vartheta| \le c\sigma^{-|\alpha|} \text{ for all multi-indices } \alpha, \tag{2.3}$$

$$0 \le g \le 1,\ g = 1 \text{ for } t \ge t^0 - \sigma^2,\ g = 0 \text{ for } t \le t^0 - \Big(\frac{3}{2}\sigma\Big)^2, \tag{2.4}$$

$$|g'(t)| \le c\sigma^{-2}.$$

Setting $\rho_{s,h}u(X) = u(x + he^s, t) - u(X)$ [3], $s = 1, 2, \dots, n$, $|h| < \frac{\sigma}{2}$, and denoting by $P_{Q(2\sigma)}$ the vector-polynomial in x, of degree ≤ 2, satisfying (1.6), from (2.1) we get in $Q(\frac{3}{2}\sigma)$

$$\rho_{s,h} a\big(H(u)\big) - \rho_{s,h}\frac{\partial u}{\partial t} = 0$$

that is

$$a\big(H(\rho_{s,h}u) + H(u)\big) - a\big(H(u)\big) - \rho_{s,h}\frac{\partial u}{\partial t} = 0,$$

from which, being $\frac{\partial}{\partial t}(\rho_{s,h}P_{Q(2\sigma)}) = 0$ and $H(\rho_{s,h}P_{Q(2\sigma)}) = 0$, we derive:

$$\begin{aligned} &\Delta(\rho_{s,h}(u - P_{Q(2\sigma)})) - \alpha\frac{\partial}{\partial t}(\rho_{s,h}(u - P_{Q(2\sigma)})) = \\ &= \Delta(\rho_{s,h}(u - P_{Q(2\sigma)})) - \alpha\Big[a(H(\rho_{s,h}(u - P_{Q(2\sigma)})) + H(u)) - a(H(u))\Big], \end{aligned} \tag{2.5}$$

where α is the positive constant that appears in the condition (A).
From (2.5), because of the condition (A), we reach:

$$\begin{aligned} &\Big\|\vartheta g\Delta(\rho_{s,h}(u - P_{Q(2\sigma)})) - \alpha\vartheta g\frac{\partial}{\partial t}(\rho_{s,h}(u - P_{Q(2\sigma)}))\Big\| = \\ &= \vartheta g\big\|\Delta(\rho_{s,h}(u - P_{Q(2\sigma)})) - \alpha\big[a(H(\rho_{s,h}(u - P_{Q(2\sigma)})) + H(u)) - a\big(H(u)\big)\big]\big\| \le \\ &\le \vartheta g\Big\{\gamma\|H(\rho_{s,h}(u - P_{Q(2\sigma)}))\|^2 + \delta\|\Delta(\rho_{s,h}(u - P_{Q(2\sigma)}))\|^2\Big\}^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

[3] $\{e^s\}_{s=1,2,\dots,n}$ is the canonic base of $\mathbb{R}^n$.

Now setting

$$\mathcal{U}(X) = \vartheta(x) g(t) \rho_{s,h}(u - P_{Q(2\sigma)}),$$

we have:

$$\mathcal{U} \in W_0^2\Big(Q\big(X^0, \frac{3}{2}\sigma\big), \mathbb{R}^N\Big), \tag{2.7}$$

$$\Delta\mathcal{U} = \vartheta g \Delta(\rho_{s,h}(u - P_{Q(2\sigma)})) + A(u - P_{Q(2\sigma)}), \tag{2.8}$$

$$H(\mathcal{U}) = \vartheta g H(\rho_{s,h}(u - P_{Q(2\sigma)})) + B(u - P_{Q(2\sigma)}), \tag{2.9}$$

$$\frac{\partial\mathcal{U}}{\partial t} = \vartheta g \frac{\partial}{\partial t}(\rho_{s,h}(u - P_{Q(2\sigma)})) + \vartheta g' \rho_{s,h}(u - P_{Q(2\sigma)}), \tag{2.10}$$

where

$$\begin{aligned} A(u - P_{Q(2\sigma)}) = {} & g\Delta\vartheta \rho_{s,h}(u - P_{Q(2\sigma)}) + \\ & + 2g \sum_{i=1}^{n} D_i\vartheta D_i(\rho_{s,h}(u - P_{Q(2\sigma)})), \end{aligned} \tag{2.11}$$

$$\begin{aligned} B(u - P_{Q(2\sigma)}) = {} & \Big\{ g D_{ij}\vartheta \rho_{s,h}(u - P_{Q(2\sigma)}) + \\ & + g D_i\vartheta D_j(\rho_{s,h}(u - P_{Q(2\sigma)})) + g D_j\vartheta D_i(\rho_{s,h}(u - P_{Q(2\sigma)}))\Big\}_{i,j=1,2,\dots,n}. \end{aligned} \tag{2.12}$$

Then, from (2.8), (2.10) and (2.6) we obtain

$$\begin{aligned} \Big\| \Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\Big\| \le {} & \Big\| \vartheta g \Delta(\rho_{s,h}(u - P_{Q(2\sigma)})) - \alpha\vartheta g \frac{\partial}{\partial t}(\rho_{s,h}(u - P_{Q(2\sigma)}))\Big\| + \\ & + \|A(u - P_{Q(2\sigma)})\| + \|\alpha\vartheta g' \rho_{s,h}(u - P_{Q(2\sigma)})\| \le \vartheta g\big\{\gamma \|H(\rho_{s,h}(u - P_{Q(2\sigma)}))\|^2 + \\ & + \delta\|\Delta(\rho_{s,h}(u - P_{Q(2\sigma)}))\|^2\big\}^{\frac{1}{2}} + \|A(u - P_{Q(2\sigma)})\| + \|\alpha\vartheta g' \rho_{s,h}(u - P_{Q(2\sigma)})\|, \end{aligned}$$

from which, by (2.8) and (2.9), it follows, $\forall\varepsilon > 0$ and for almost every $X \in Q(\frac{3}{2}\sigma)$:

$$\begin{aligned} \Big\| \Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\Big\|^2 \le {} & (1+\varepsilon)\vartheta^2 g^2\big\{\gamma\|H(\rho_{s,h}(u - P_{Q(2\sigma)}))\|^2 + \\ & + \delta\|\Delta(\rho_{s,h}(u - P_{Q(2\sigma)}))\|^2\big\} + c(\varepsilon)\big\{\|A(u - P_{Q(2\sigma)})\|^2 + \\ & + \|\alpha\vartheta g' \rho_{s,h}(u - P_{Q(2\sigma)})\|^2\big\} \le (1+\varepsilon)^2\big\{\gamma\|H(\mathcal{U})\|^2 + \delta\|\Delta\mathcal{U}\|^2\big\} + \\ & + c(\varepsilon, \alpha, \gamma, \delta)\{\|A(u - P_{Q(2\sigma)})\|^2 + \|B(u - P_{Q(2\sigma)})\|^2 + \vartheta^2 g'^2 \|\rho_{s,h}(u - P_{Q(2\sigma)})\|^2\}. \end{aligned} \tag{2.13}$$

Integrating (2.13) on $Q(\frac{3}{2}\sigma)$, using (2.7) and taking into consideration Lemma 2.4 of [3], we obtain:

$$\Big[1-(1+\varepsilon)^2\delta\Big]\int_{Q(\frac{3}{2}\sigma)}\Big\|\Delta\mathcal{U}-\alpha\frac{\partial\mathcal{U}}{\partial t}\Big\|^2\,dX\le$$

$$\le(1+\varepsilon)^2\gamma\int_{Q(\frac{3}{2}\sigma)}\|H(\mathcal{U})\|^2\,dX+$$

$$+c(\varepsilon,\alpha,\gamma,\delta)\int_{Q(\frac{3}{2}\sigma)}\big(\|A(u-P_{Q(2\sigma)})\|^2+\|B(u-P_{Q(2\sigma)})\|^2+$$

$$+\vartheta^2g'^2\|\rho_{s,h}(u-P_{Q(2\sigma)}\big)\|^2)\,dX,$$

from which, by Lemma 2.3 of [3] and for each $\varepsilon\in\big]\,0,\frac{1}{\sqrt{\delta}}-1\big[$, we deduce:

$$\Big[1-(1+\varepsilon)^2\delta\Big]\int_{Q(\frac{3}{2}\sigma)}\Big(\|H(\mathcal{U})\|^2+\alpha^2\Big\|\frac{\partial\mathcal{U}}{\partial t}\Big\|^2\Big)\,dX\le$$

$$\le(1+\varepsilon)^2\gamma\int_{Q(\frac{3}{2}\sigma)}\Big(\|H(\mathcal{U})\|^2+\alpha^2\Big\|\frac{\partial\mathcal{U}}{\partial t}\Big\|^2\Big)\,dX+$$

$$+c(\varepsilon,\alpha,\gamma,\delta)\int_{Q(\frac{3}{2}\sigma)}\big(\|A(u-P_{Q(2\sigma)})\|^2+\|B(u-P_{Q(2\sigma)})\|^2+$$

$$+\vartheta^2g'^2\|\rho_{s,h}(u-P_{Q(2\sigma)}\big)\|^2)\,dX$$

and hence, for ε chosen in the interval $\big]\,0,\frac{1}{\sqrt{\gamma+\delta}}-1\big[$, we get ([4]):

$$\int_{Q(\frac{3}{2}\sigma)}\Big(\|H(\mathcal{U})\|^2+\alpha^2\Big\|\frac{\partial\mathcal{U}}{\partial t}\Big\|^2\Big)\,dX\le\tag{2.14}$$

$$\le c\int_{Q(\frac{3}{2}\sigma)}\big(\|A(u-P_{Q(2\sigma)})\|^2+\|B(u-P_{Q(2\sigma)})\|^2+\sigma^{-4}\|\rho_{s,h}(u-P_{Q(2\sigma)})\|^2\big)\,dX.$$

From (2.14), taking into account (2.9) and (2.10), it follows:

$$\int_{Q(\sigma)}\Big(\|H(\rho_{s,h}u)\|^2+\alpha^2\Big\|\frac{\partial}{\partial t}(\rho_{s,h}u)\Big\|^2\Big)\,dX\le$$

([4]) Let us remember that $0\le\vartheta\le1$ and that $|g'|\le c\sigma^{-2}$.
In the next estimate, c denotes a constant which depends on $\alpha,\gamma,\delta,\varepsilon$ and on the constant that appears in the last of the estimates (2.4).

$$\leq \int_{Q(\frac{3}{2}\sigma)} \vartheta^2 g^2 \Big(\|H(\rho_{s,h} u)\|^2 + \alpha^2 \Big\| \frac{\partial}{\partial t}(\rho_{s,h} u) \Big\|^2 \Big)\, dX \leq$$

$$\leq c \Big\{ \int_{Q(\frac{3}{2}\sigma)} \|A(u - P_{Q(2\sigma)})\|^2\, dX + \int_{Q(\frac{3}{2}\sigma)} \|B(u - P_{Q(2\sigma)})\|^2\, dX +$$

$$+ \sigma^{-4} \int_{Q(\frac{3}{2}\sigma)} \|\rho_{s,h}(u - P_{Q(2\sigma)})\|^2\, dX \Big\},$$

from which, in virtue of (2.11), (2.12), (2.3) and (2.4), we get:

$$\int_{Q(\sigma)} \Big(\|H(\rho_{s,h} u)\|^2 + \alpha^2 \Big\| \frac{\partial}{\partial t}(\rho_{s,h} u) \Big\|^2 \Big) dX \leq \tag{2.15}$$

$$\leq c\sigma^{-4} \int_{Q(\frac{3}{2}\sigma)} \|\rho_{s,h}(u - P_{Q(2\sigma)})\|^2\, dX + c\sigma^{-2} \int_{Q(\frac{3}{2}\sigma)} \|\rho_{s,h} D(u - P_{Q(2\sigma)})\|^2\, dX.$$

We shall now evaluate the integrals that appear in the right hand side of (2.15) using Lemma 2.I of [7]. We obtain, for $|h| < \frac{\sigma}{2}$ and $s = 1, 2, \dots, n$:

$$\int_{Q(\sigma)} \Big(\|\rho_{s,h} H(u)\|^2 + \alpha^2 \Big\| \rho_{s,h} \frac{\partial u}{\partial t} \Big\|^2 \Big)\, dX \leq \tag{2.16}$$

$$\leq c\sigma^{-2} |h|^2 \Big\{ \sigma^{-2} \int_{Q(2\sigma)} \|D(u - P_{Q(2\sigma)})\|^2\, dX + \int_{Q(2\sigma)} \|H(u - P_{Q(2\sigma)})\|^2\, dX \Big\}.$$

From (2.16), by Lemma 3.1 of [9], it follows that

$$H(u) \in L^2(t^0 - \sigma^2, t^0, H^1(B(x^0, \sigma), \mathbb{R}^{n^2 N})),$$

$$\frac{\partial u}{\partial t} \in L^2(t^0 - \sigma^2, t^0, H^1(B(x^0, \sigma), \mathbb{R}^N))$$

and also the following estimate holds:

$$\int_{Q(\sigma)} \Big(\|D\big(H(u)\big)\|^2 + \Big\| D\big(\frac{\partial u}{\partial t}\big) \Big\|^2 \Big)\, dX \leq$$

$$\leq c\sigma^{-2} \Big\{ \sigma^{-2} \int_{Q(2\sigma)} \|D(u - P_{Q(2\sigma)})\|^2 dX + \int_{Q(2\sigma)} \|H(u - P_{Q(2\sigma)})\|^2 dX \Big\}.$$

Then (1.4), (1.5) and Theorem 1.1 are proved. □

Theorem 1.1 ensures that, if $u \in W^2(Q, \mathbb{R}^N)$ is a solution in Q of the system (2.1), fixed the cylinder $Q(2\sigma) = Q(X^0, 2\sigma) \subset\subset Q$, it follows

$$Du \in W^2(Q(2\sigma), \mathbb{R}^{nN}). \tag{2.17}$$

On the other hand, if $P_{Q(2\sigma)}$ is the vector-polynomial in x, of degree ≤ 2, such that

$$\int_{Q(2\sigma)} D^\alpha(u - P_{Q(2\sigma)})dX = 0, \ \forall\alpha : |\alpha| \leq 2,$$

$DP_{Q(2\sigma)}$ turns out to be the vector-polynomial in x, of degree ≤ 1, such that

$$\int_{Q(2\sigma)} D^\alpha(Du - DP_{Q(2\sigma)})dX = 0, \ \forall\alpha : |\alpha| \leq 1.$$

From this remark and taking into account (2.17), it follows, by Lemma 2.2 of [8] (written for 2σ, Du and $DP_{Q(2\sigma)}$ instead of σ, u and $P_{Q(X^0,\sigma)}$, respectively):

$$\begin{aligned}\sigma^{-2}\int_{Q(2\sigma)} \|D(u - P_{Q(2\sigma)})\|^2\, dX + \int_{Q(2\sigma)} \|H(u - P_{Q(2\sigma)})\|^2\, dX \leq \\ \leq c\left[\int_{Q(2\sigma)} \left(\|H(Du)\|^{\frac{2(n+2)}{n+4}} + \left\|\frac{\partial}{\partial t}(Du)\right\|^{\frac{2(n+2)}{n+4}}\right)dX\right]^{\frac{n+4}{n+2}},\end{aligned} \tag{2.18}$$

where $\sigma \in (0, 1)$ and the constant c does not depend on σ.
Then, under the assumptions of Theorem 1.1, from (1.5) and (2.18), we deduce, $\forall\, Q(2\sigma) \subset\subset Q$ with $\sigma \in (0, 1)$:

$$\begin{aligned}&\fint_{Q(\sigma)} \left(\|H(Du)\|^2 + \left\|\frac{\partial(Du)}{\partial t}\right\|^2\right) dX \leq \\ &\leq c\left[\fint_{Q(2\sigma)} \left(\|H(Du)\|^{\frac{2(n+2)}{n+4}} + \left\|\frac{\partial(Du)}{\partial t}\right\|^{\frac{2(n+2)}{n+4}}\right) dX\right]^{\frac{n+4}{n+2}},\end{aligned}$$

where the constant c does not depend on σ.
From this, by a classical lemma of Gehring-Giaquinta-G. Modica (see, for example, [8], Lemma 3.3), we derive that $\exists\, \tilde{q} > 2$ such that, $\forall q \in (2, \tilde{q})$,

$$Du \in W^q_{loc}(Q, \mathbb{R}^{nN})$$

and, $\forall\, Q(2\sigma) \subset\subset Q$, with $\sigma \in (0, 1)$

$$\begin{aligned}&\left[\fint_{Q(\sigma)} \left(\|H(Du)\|^q + \left\|\frac{\partial(Du)}{\partial t}\right\|^q\right) dX\right]^{\frac{1}{q}} \leq \\ &\leq c\left[\fint_{Q(2\sigma)} \left(\|H(Du)\|^2 + \left\|\frac{\partial(Du)}{\partial t}\right\|^2\right) dX\right]^{\frac{1}{2}}.\end{aligned} \tag{2.19}$$

Now let us give the proof of Theorem 1.2. The proof is similar to that one used in [3], Theorem 1.1 (see also [6], Theorem 2.1). We present the proof for the reader's convenience. Having fixed $\varphi \in L^2(Q, \mathbb{R}^N)$ and $u \in W^2(Q, \mathbb{R}^N)$ we must prove that the corresponding problem (1.7) admits a unique solution w and that the estimate (1.8) holds. The condition (1.3) ensures that the operator

$$A(w) = a\big(H(w) + H(u)\big) - \frac{\partial w}{\partial t}$$

associates to each $w \in W_0^2(Q, \mathbb{R}^N)$ an element of $L^2(Q, \mathbb{R}^N)$:

$$A(w) : W_0^2(Q, \mathbb{R}^N) \to L^2(Q, \mathbb{R}^N).$$

On the other hand it is well known that the linear operator

$$B(w) = \Delta w - \alpha \frac{\partial w}{\partial t} \ (^5)$$

is an isomorphism $W_0^2(Q, \mathbb{R}^N) \to L^2(Q, \mathbb{R}^N)$.
Let us show that $A(w)$ is "near" to the operator $B(w)$ (^{6}). For each $w_1, w_2 \in W_0^2(Q, \mathbb{R}^N)$, we have, by condition (A) and in view of the Lemmas 2.3 and 2.4 of [3]:

$$\begin{aligned}
&\|B(w_1) - B(w_2) - \alpha[A(w_1) - A(w_2)]\|^2_{L^2(Q,\mathbb{R}^N)} = \\
&= \int_Q \big\|\Delta(w_1 - w_2) - \alpha\big[a\big(H(w_1 - w_2) + H(w_2) + H(u)\big) - \\
&- a\big(H(w_2) + H(u)\big)\big]\big\|^2 \, dX \le \gamma \int_Q \|H(w_1 - w_2)\|^2 \, dX + \\
&+\delta \int_Q \|\Delta(w_1 - w_2)\|^2 \, dX \le \gamma \int_Q \left[\|H(w_1 - w_2)\|^2 + \alpha^2 \left\|\frac{\partial(w_1 - w_2)}{\partial t}\right\|^2\right] dX + \\
&+\delta \int_Q \|\Delta(w_1 - w_2)\|^2 \, dX \le (\gamma + \delta) \int_Q \left\|\Delta(w_1 - w_2) - \alpha \frac{\partial(w_1 - w_2)}{\partial t}\right\|^2 dX = \\
&= (\gamma + \delta)\|B(w_1) - B(w_2)\|^2_{L^2(Q,\mathbb{R}^N)} ,
\end{aligned}$$

from which it follows

$$\|B(w_1) - B(w_2) - \alpha[A(w_1) - A(w_2)]\|_{L^2(Q,\mathbb{R}^N)} \le K \|B(w_1) - B(w_2)\|_{L^2(Q,\mathbb{R}^N)},$$

(^{5}) α is the positive constant that appears in the condition (A).
(^{6}) In the sense of Definition 1 of [4].

with $K = \sqrt{\gamma + \delta}$; hence the operator $A(w)$ is near to the operator $B(w)$. Then Theorem 2 of [4] ensures that the Cauchy-Dirichlet problem (1.7) has a unique solution $w \in W_0^2(Q, \mathbb{R}^N)$ and that this solution fulfills the estimate:

$$\|B(w)\|_{L^2(Q,\mathbb{R}^N)} \leq \frac{\alpha}{1 - \sqrt{\gamma + \delta}} \, \|\varphi - A(0)\|_{L^2(Q,\mathbb{R}^N)}. \tag{2.20}$$

From (2.20), by Lemma 2.3 of [3], (1.8) it follows. □

3. Interior fundamental estimates.

Let $u \in W^2(Q, \mathbb{R}^N)$ be a solution in Q of the basic system (1.1). The following fundamental estimates for $H(Du)$, $\frac{\partial(Du)}{\partial t}$, $H(u)$ and $\frac{\partial u}{\partial t}$ hold:

Theorem 3.1. *If the vector* $a(\xi)$ *satisfies the conditions* (1.2) *and* (A)*, then,* $\forall\, Q(\sigma) \subset\subset Q$*, with* $\sigma < 2$*,* $\forall \tau \in (0, 1)$ *and* $\forall q \in (2, \tilde{q})$ (7)*, we have:*

$$\int_{Q(\tau\sigma)} \left(\|H(Du)\|^2 + \left\| \frac{\partial(Du)}{\partial t} \right\|^2 \right) dX \leq \tag{3.1}$$
$$\leq c\tau^{(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \left(\|H(Du)\|^2 + \left\| \frac{\partial(Du)}{\partial t} \right\|^2 \right) dX,$$

where the constant c does not depend on σ *and* τ*.*

Proof. Fixed $Q(\sigma) \subset\subset Q$, with $\sigma < 2$, $\tau \in (0, \frac{1}{2})$, in virtue of the L^q_{loc}-result showed in Section 2, we have:

$$Du \in W^q(Q(\sigma), \mathbb{R}^{nN}), \ \forall q \in (2, \tilde{q});$$

then, by Hölder's inequality, we get

$$\int_{Q(\tau\sigma)} \left(\|H(Du)\|^2 + \left\| \frac{\partial(Du)}{\partial t} \right\|^2 \right) dX \leq$$
$$\leq c\left[\int_{Q(\tau\sigma)} \left(\|H(Du)\|^q + \left\| \frac{\partial(Du)}{\partial t} \right\|^q \right) dX \right]^{\frac{2}{q}} (\tau\sigma)^{(n+2)(1-\frac{2}{q})} \leq$$
$$\leq c\tau^{(n+2)(1-\frac{2}{q})} \sigma^{n+2} \left[⨍_{Q(\frac{\sigma}{2})} \left(\|H(Du)\|^q + \left\| \frac{\partial(Du)}{\partial t} \right\|^q \right) dX \right]^{\frac{2}{q}}$$

(7) $\tilde{q}$ is the constant (> 2) which appears in (2.19).

from which, in virtue of (2.19), the estimate (3.1) follows with $\tau \in (0, \frac{1}{2})$. The estimate is trivially true for $\frac{1}{2} \leq \tau < 1$. $\square$

Theorem 3.2. *If the vector $a(\xi)$ satisfies the conditions* (1.2) *and* (A)*, then, $\forall\, Q(\sigma) \subset Q$, with $\sigma < 2$, $\forall\, \tau \in (0, 1)$ and $\forall\, q \in (2, \min(\tilde{q}, n+2))$* ([7])*, we have:*

$$\int_{Q(\tau\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \leq \tag{3.2}$$

$$\leq c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX,$$

where the constant c does not depend on σ and τ.

Proof. Fixed $Q(\sigma) \subset Q$, with $\sigma < 2$, and $q \in (2, \min(\tilde{q}, n+2))$, for $0 < \tau < \tau' < \frac{1}{2}$, by means of Lemma 2.II of [5] (written for Du instead of u), we get ([8]):

$$\int_{Q(\tau\sigma)} \|H(u)\|^2 \, dX \leq 2 \int_{Q(\tau\sigma)} \left\| \big(H(u)\big)_{Q(\tau'\sigma)} \right\|^2 dX +$$

$$+ 2 \int_{Q(\tau\sigma)} \left\| H(u) - \big(H(u)\big)_{Q(\tau'\sigma)} \right\|^2 dX \leq c \left(\frac{\tau}{\tau'} \right)^{n+2} \int_{Q(\tau'\sigma)} \|H(u)\|^2 \, dX +$$

$$+ c(\tau'\sigma)^2 \int_{Q(\tau'\sigma)} \left(\|H(Du)\|^2 + \left\| \frac{\partial(Du)}{\partial t} \right\|^2 \right) dX,$$

from which, using (3.1), it follows:

$$\int_{Q(\tau\sigma)} \|H(u)\|^2 \, dX \leq c \left(\frac{\tau}{\tau'} \right)^{n+2} \int_{Q(\tau'\sigma)} \|H(u)\|^2 \, dX +$$

$$+ c\sigma^2 \tau'^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\frac{\sigma}{2})} \left(\|H(Du)\|^2 + \left\| \frac{\partial(Du)}{\partial t} \right\|^2 \right) dX.$$

([8]) If $E \subset \mathbb{R}^{n+1}$ is a measurable set with positive measure and $f \in L^1(E, \mathbb{R}^k)$, we set:

$$f_E = \fint_E f \, dX = \frac{1}{\operatorname{meas} E} \int_E f \, dX.$$

From this, taking into account Lemma 1.I, p.7 of [1], being $2+(n+2)(1-\frac{2}{q}) < n+2$, we get:

$$\int_{Q(\tau\sigma)} \|H(u)\|^2\, dX \leq c\left(\frac{\tau}{\tau'}\right)^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\tau'\sigma)} \|H(u)\|^2\, dX + \\ + c\sigma^2 \tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\frac{\sigma}{2})} \Big(\|H(Du)\|^2 + \Big\|\frac{\partial(Du)}{\partial t}\Big\|^2\Big)\, dX$$

and hence, taking the limit for $\tau' \to \frac{1}{2}$, we derive, $\forall 0 < \tau < \frac{1}{2}$:

$$\int_{Q(\tau\sigma)} \|H(u)\|^2\, dX \leq c\tau^{2+(n+2)(1-\frac{2}{q})}\Big\{ \int_{Q(\sigma)} \|H(u)\|^2\, dX + \\ + \sigma^2 \int_{Q(\frac{\sigma}{2})} \Big(\|H(Du)\|^2 + \Big\|\frac{\partial(Du)}{\partial t}\Big\|^2\Big)\, dX\Big\}. \tag{3.3}$$

On the other hand we have the estimates of Caccioppoli (1.5) and of Poincaré (see Lemma 2.II of [5]); then applying these estimates we get:

$$\sigma^2 \int_{Q(\frac{\sigma}{2})} \Big(\|H(Du)\|^2 + \Big\|\frac{\partial(Du)}{\partial t}\Big\|^2\Big)\, dX \leq \\ \leq c\left\{\sigma^{-2}\int_{Q(\sigma)} \big\|D(u-P_{Q(\sigma)})\big\|^2\, dX + \int_{Q(\sigma)} \big\|H(u-P_{Q(\sigma)})\big\|^2\, dX\right\} \leq \\ \leq c\Big\{\sigma^{-2}\int_{Q(\sigma)} \big\|Du-(Du)_{Q(\sigma)}\big\|^2\, dX + \int_{Q(\sigma)} \|H(u)\|^2\, dX + \\ + \int_{Q(\sigma)} \big\|H(u)-\big(H(u)\big)_{Q(\sigma)}\big\|^2\, dX\Big\} \leq c\int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX + \\ + c\int_{Q(\sigma)} \big\|H(u)-\big(H(u)\big)_{Q(\sigma)}\big\|^2\, dX \leq c\int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX, \tag{3.4}$$

where $P_{Q(\sigma)}$ is the vector-polynomial in x, of degree ≤ 2, such that

$$\int_{Q(\sigma)} D^\alpha(u-P_{Q(\sigma)})\, dX = 0,\ \forall\alpha : |\alpha| \leq 2.$$

Hence from (3.3) and (3.4) we get, $\forall 0 < \tau < \frac{1}{2}$

$$\int_{Q(\tau\sigma)} \|H(u)\|^2\, dX \leq c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX. \tag{3.5}$$

Now let us observe that, being $\frac{\partial u}{\partial t} = a\big(H(u)\big)$ in Q, using estimate (1.3), we obtain:

$$\left\| \frac{\partial u}{\partial t} \right\|^2 = \left\| a\big(H(u)\big) \right\|^2 \leq c \|H(u)\|^2$$

and hence

$$\int_{Q(\tau\sigma)} \left(\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \leq c \int_{Q(\tau\sigma)} \|H(u)\|^2 \, dX. \tag{3.6}$$

From (3.5) and (3.6) the assertion follows for $0 < \tau < \frac{1}{2}$. Finally, the estimate (3.2) is trivially true for $\frac{1}{2} \leq \tau < 1$. $\square$

The estimate (3.2) ensures that, $\forall q \in \big(2, \min(\tilde{q}, n+2)\big)$:

$$H(u) \in L^{2,2+(n+2)(1-\frac{2}{q})}_{\text{loc}}(Q, \mathbb{R}^{n^2 N}) \tag{3.7}$$

and

$$\frac{\partial u}{\partial t} \in L^{2,2+(n+2)(1-\frac{2}{q})}_{\text{loc}}(Q, \mathbb{R}^{N}); \tag{3.8}$$

therefore, in virtue of Lemma 2.II by [5]:

$$Du \in \mathcal{L}^{2,4+(n+2)(1-\frac{2}{q})}_{\text{loc}}(Q, \mathbb{R}^{nN}), \ \ \forall q \in \big(2, \min(\tilde{q}, n+2)\big). \tag{3.9}$$

Now if $n < 2\tilde{q}-2$ (and in particular if $n = 2$), there exists $q \in \big(2, \min(\tilde{q}, n+2)\big)$ such that $4 + (n+2)(1 - \frac{2}{q}) > n + 2$ and hence, by (3.9)

$$Du \ \textit{is Hölder-continuous in} \ Q. \tag{3.10}$$

We also obtain from Lemma 2.I of [5] and conditions (3.8) and (3.9)

$$u \in \mathcal{L}^{2,6+(n+2)(1-\frac{2}{q})}_{\text{loc}}(Q, \mathbb{R}^{N}), \ \ \forall q \in \big(2, \min(\tilde{q}, n+2)\big),$$

and hence, if $n < 3\tilde{q} - 2$ (and in particular if $n \leq 4$), we derive

$$u \ \textit{is Hölder-continuous in} \ Q. \tag{3.11}$$

The results (3.10) and (3.11) are similar to those obtained by S. Campanato in Section 5 of [3].

4. $\mathcal{L}^{2,\lambda}$ -regularity for systems of type (1.9).

Let $f : Q \to \mathbb{R}^N$ be a vector of class $L^2(Q, \mathbb{R}^N)$ and $u \in W^2(Q, \mathbb{R}^N)$ a solution in Q of the parabolic system

$$a\big(H(u)\big) - \frac{\partial u}{\partial t} = f(X), \tag{4.1}$$

with $a(\xi)$ vector of $\mathbb{R}^N$, continuous onto $\mathbb{R}^{n^2 N}$, satisfying the conditions (1.2) and (A).

Let us show the following

Lemma 4.1. *For each cylinder $Q(\sigma) \subset Q$, with $\sigma < 2$, $\forall \tau \in (0, 1)$ and $\forall q \in \big(2, \min(\tilde{q}, n+2)\big)$ (7), one has:*

$$\int_{Q(\tau\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX \le$$
$$\le c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX + c \int_{Q(\sigma)} \|f\|^2\, dX,$$

where the constant c does not depend on σ and τ.

Proof. Fixed $Q(\sigma) \subset Q$, with $\sigma < 2$, let w be the solution of the Cauchy-Dirichlet problem:

$$\begin{cases} w \in W_0^2(Q(\sigma), \mathbb{R}^N) \\ a\big(H(w) + H(u)\big) - \dfrac{\partial w}{\partial t} = \dfrac{\partial u}{\partial t} \quad \text{in } Q(\sigma) \end{cases} \quad (^9). \tag{4.2}$$

Setting in $Q(\sigma)$ $v = w + u$, we have: $v \in W^2(Q(\sigma), \mathbb{R}^N)$ and

$$a\big(H(v)\big) - \frac{\partial v}{\partial t} = 0 \quad \text{in } Q(\sigma). \tag{4.3}$$

We have for v the fundamental estimate (3.2):

$$\int_{Q(\tau\sigma)} \Big(\|H(v)\|^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)\, dX \le \tag{4.4}$$
$$\le c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \Big(\|H(v)\|^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)\, dX,$$

(9) Theorem 1.2 ensures the existence of an unique solution of the problem (4.2).

$\forall \tau \in (0, 1)$ and $\forall q \in \big(2, \min(\tilde{q}, n + 2)\big)$.

On the other hand from (1.8), it follows

$$\int_{Q(\sigma)} \Big(\|H(w)\|^2 + \Big\| \frac{\partial w}{\partial t} \Big\|^2 \Big)\, dX \le \le c(\alpha, \gamma, \delta) \int_{Q(\sigma)} \Big\| \frac{\partial u}{\partial t} - a\big(H(u)\big) \Big\|^2 \, dX \tag{4.5}$$

and also, in virtue of (4.1):

$$\int_{Q(\sigma)} \Big(\|H(w)\|^2 + \Big\| \frac{\partial w}{\partial t} \Big\|^2 \Big)\, dX \le c(\alpha, \gamma, \delta) \int_{Q(\sigma)} \|f\|^2 \, dX. \tag{4.6}$$

From (4.4) and taking into account that $u = v - w$, it follows, $\forall \tau \in (0, 1)$:

$$\begin{aligned}
&\int_{Q(\tau\sigma)} \Big(\|H(u)\|^2 + \Big\| \frac{\partial u}{\partial t} \Big\|^2 \Big)\, dX \le \\
&\le c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \Big(\|H(v)\|^2 + \Big\| \frac{\partial v}{\partial t} \Big\|^2 \Big)\, dX + \\
&\quad + 2 \int_{Q(\sigma)} \Big(\|H(w)\|^2 + \Big\| \frac{\partial w}{\partial t} \Big\|^2 \Big)\, dX \le \\
&\le c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\| \frac{\partial u}{\partial t} \Big\|^2 \Big)\, dX + \\
&\quad + c \int_{Q(\sigma)} \Big(\|H(w)\|^2 + \Big\| \frac{\partial w}{\partial t} \Big\|^2 \Big)\, dX
\end{aligned}$$

from which, by (4.6), we deduce:

$$\begin{aligned}
&\int_{Q(\tau\sigma)} \Big(\|H(u)\|^2 + \Big\| \frac{\partial u}{\partial t} \Big\|^2 \Big)\, dX \le \\
&\le c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\| \frac{\partial u}{\partial t} \Big\|^2 \Big)\, dX + c \int_{Q(\sigma)} \|f\|^2 \, dX.
\end{aligned}$$

$\square$

Lemma 4.1 enables us to prove the following

Theorem 4.1. *If $f \in \mathcal{L}^{2,\mu}(Q, \mathbb{R}^N)$, $0 < \mu < \tilde{\lambda} = \min\{2 + (n+2)(1 - \frac{2}{\tilde{q}}), n + 2\}$, if $u \in W^2(Q, \mathbb{R}^N)$ is a solution of the system*

$$a\big(H(u)\big) - \frac{\partial u}{\partial t} = f(X) \quad \text{in } Q$$

and if the vector $a(\xi)$ satisfies the conditions (1.2) *and* (A)*, then*

$$Du \in \mathcal{L}^{2,\mu+2}_{\mathrm{loc}}(Q, \mathbb{R}^{nN}) \tag{4.7}$$

and

$$u \in \mathcal{L}^{2,\mu+4}_{\mathrm{loc}}(Q, \mathbb{R}^{N}). \tag{4.8}$$

Proof. Fixed $Q(\sigma) \subset Q$, with $\sigma < 2$, for each $\tau \in (0, 1)$ and for each $q \in \big(2, \min(\tilde{q}, n+2)\big)$, in virtue of Lemma 4.1, we get:

$$\begin{aligned}(4.9)\qquad &\int_{Q(\tau\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX \le \\ &\le c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX + c\int_{Q(\sigma)} \|f\|^2\, dX\end{aligned}$$

and also, by assumption $f \in \mathcal{L}^{2,\mu}(Q, \mathbb{R}^N)$:

$$\begin{aligned}(4.10)\qquad &\int_{Q(\tau\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX \le \\ &\le c\tau^{2+(n+2)(1-\frac{2}{q})} \int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX + c\sigma^{\mu}\|f\|^2_{\mathcal{L}^{2,\mu}(Q,\mathbb{R}^N)}.\end{aligned}$$

Now, choosing $q \in \big(2, \min(\tilde{q}, n+2)\big)$ in such a way that $2 + (n+2)(1 - \frac{2}{q}) > \mu$, by (4.10) (written for this value of q) and Lemma 1.I, p. 7 of [1], we obtain:

$$\begin{aligned}(4.11)\qquad &\int_{Q(\tau\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX \le \\ &\le c\tau^{\mu}\Big\{\int_{Q(\sigma)} \Big(\|H(u)\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\, dX + \sigma^{\mu}\|f\|^2_{\mathcal{L}^{2,\mu}(Q,\mathbb{R}^N)}\Big\}.\end{aligned}$$

The estimate (4.11) ensures that

$$H(u) \in L^{2,\mu}_{\mathrm{loc}}(Q, \mathbb{R}^{n^2N}) \tag{4.12}$$

and

$$\frac{\partial u}{\partial t} \in L^{2,\mu}_{\mathrm{loc}}(Q, \mathbb{R}^N), \tag{4.13}$$

and hence (4.7), by Lemma 2.II of [5].
Finally the condition (4.8) is a consequence of (4.7), (4.13) and Lemma 2.I of [5]. □

If $n < 2\tilde{q} - 2$ (and in particular if $n = 2$) and if $f \in \mathcal{L}^{2,\mu}(Q, \mathbb{R}^N)$ with $\mu \in (n, \tilde{\lambda})$, in virtue of (4.7), we get:

$$Du \in \mathcal{L}^{2,\mu+2}_{\mathrm{loc}}(Q, \mathbb{R}^{nN}), \quad \text{with } \mu + 2 > n + 2,$$

and hence

$$Du \textit{ is Hölder-continuous in } Q.$$

Similarly, if $n < 3\tilde{q} - 2$ (and in particular if $n \leq 4$) and if $f \in \mathcal{L}^{2,\mu}(Q, \mathbb{R}^N)$ with $\mu \in (n-2, \tilde{\lambda})$, in virtue of (4.8) we obtain:

$$u \in \mathcal{L}^{2,\mu+4}_{\mathrm{loc}}(Q, \mathbb{R}^N), \quad \text{with } \mu + 4 > n + 2,$$

and hence

$$u \textit{ is Hölder-continuous in } Q.$$

5. $\mathcal{L}^{2,\lambda}$-regularity for systems of type (1.10).

Let $u \in W^2(Q, \mathbb{R}^N)$ be a solution of the system

$$a\big(H(u)\big) - \frac{\partial u}{\partial t} = b(X, u, Du) \quad \text{in } Q, \tag{5.1}$$

where $a(\xi)$ is a vector of $\mathbb{R}^N$, continuous onto $\mathbb{R}^{n^2 N}$ and satisfying the conditions (1.2) and (A) and $b(X, u, p)$ is a vector of $\mathbb{R}^N$, measurable in X, continuous in (u, p) and satisfying the condition

(5.2) *there exists a constant c such that, $\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$ and for almost every $X \in Q$:*

$$\|b(X, u, p)\| \leq c\Big(1 + \|u\| + \|p\|\Big).$$

Lemmas 2.1 of [8] and 2.II of [5] and Theorem 3.1 of [8] ensure that

$$u \in \mathcal{L}^{2,4+(n+2)(1-\frac{2}{q})}_{\mathrm{loc}}(Q,\mathbb{R}^N),\ Du \in \mathcal{L}^{2,2+(n+2)(1-\frac{2}{q})}_{\mathrm{loc}}(Q,\mathbb{R}^{nN}),\ \ \forall q \in (2,\bar{q})(^{10}),$$

and, hence, u and $D_i u$, $i = 1,2,\dots,n$, belong to $\mathcal{L}^{2,\mu}(Q^*,\mathbb{R}^N)$, $\forall \mu \in \big(0, 2+(n+2)(1-\frac{2}{\bar{q}})\big)$ and $\forall Q^* \subset\subset Q$. From this, taking into account condition (5.2), it follows that the vector $f(X) = b(X,u,Du) \in \mathcal{L}^{2,\mu}(Q^*,\mathbb{R}^N)$, $\forall \mu \in \big(0, 2+(n+2)(1-\frac{2}{\tilde{q}})\big)$ and $\forall Q^* \subset\subset Q$.
Then Theorem 4.1 implies

$$Du \in \mathcal{L}^{2,\mu+2}_{\mathrm{loc}}(Q,\mathbb{R}^{nN})\ ,\ u \in \mathcal{L}^{2,\mu+4}_{\mathrm{loc}}(Q,\mathbb{R}^N),\ \ \forall \mu \in (0,\lambda^*), \tag{5.3}$$

where $\lambda^* = \min\Big\{2+(n+2)(1-\frac{2}{\bar{q}}),\ 2+(n+2)(1-\frac{2}{\tilde{q}}),\ n+2\Big\} = \min\Big\{2+(n+2)(1-\frac{2}{q^*}), n+2\Big\}$, $q^* = \min(\bar{q},\tilde{q})$.
Now if $n < 2q^* - 2$, it results $n < \lambda^*$. Then denoting by μ' a number of the interval (n,λ^*), from the first statement of (5.3) it follows

$$Du \in \mathcal{L}^{2,\mu'+2}_{\mathrm{loc}}(Q,\mathbb{R}^{nN}),$$

and hence, being $\mu' + 2 > n+2$, Du is Hölder-continuous in Q. In particular

$$Du \ \textit{ is Hölder-continuous in } \ Q \ \text{ if } \ n = 2.$$

If $n < 3q^* - 2$, then $n - 2 < \lambda^*$ and hence, fixed $\mu'' \in (n-2,\lambda^*)$, from the second statement of (5.3), it follows

$$u \in \mathcal{L}^{2,\mu''+4}_{\mathrm{loc}}(Q,\mathbb{R}^N),$$

from which, being $\mu'' + 4 > n+2$, the Hölder-continuity of u in Q follows. In particular the vector

$$u \ \textit{ is Hölder-continuous in } \ Q \ \text{ if } \ n \le 4.$$

(10) $\bar{q}$ is the constant (> 2) which appears in the Theorem 3.1 of [8]. In [8] Lemma 2.1 and Theorem 3.1 are proved in the hypothesis $n > 2$. These results are true also for $n = 2$.

REFERENCES

[1] S. Campanato, *Sistemi ellittici in forma divergenza. Regolarità all'interno,* Quaderni Scuola Norm. Sup. Pisa, 1980.

[2] S. Campanato, *Non variational basic elliptic systems of second order,* Rend. Sem. Mat. Fis. Milano, 60 (1990), pp. 113-131.

[3] S. Campanato, *Non variational basic parabolic systems of second order,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (9) 2 (1991), pp. 129-136.

[4] S. Campanato, *On the condition of nearness between operators,* Ann. Mat. Pura Appl., (4) 167 (1994), pp. 243-256.

[5] P. Cannarsa, *Second order non variational parabolic systems,* Boll. U.M.I., (5) 18-C (1981), pp. 291-315.

[6] M.S. Fanciullo, $\mathcal{L}^{2,\lambda}$ *regularity for second order non linear non variational elliptic systems,* Le Matematiche, 50 (1995), pp. 163-172.

[7] L. Fattorusso, *Sulla differenziabilità delle soluzioni di sistemi parabolici non lineari del secondo ordine ad andamento quadratico,* Boll. U.M.I., (7) 1-B (1987), pp. 741-764.

[8] M. Marino - A. Maugeri, *Second order non linear non variational parabolic systems,* Rend. Mat., (7) 13 (1993), pp. 499-527.

[9] M. Marino - A. Maugeri, *Differentiability of weak solutions of nonlinear parabolic systems with quadratic growth,* Le Matematiche, 50 (1995), pp. 361-377.

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